Pancyclicity of Hamiltonian and highly connected graphs

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Abstract

A graph G on n vertices is *Hamiltonian* if it contains a cycle of length n and *pancyclic* if it contains cycles of length ℓ for all $3 \leq \ell \leq n$. Write $\alpha(G)$ for the *independence number* of G, i.e. the size of the largest subset of the vertex set that does not contain an edge, and $\kappa(G)$ for the (vertex) *connectivity*, i.e. the size of the smallest subset of the vertex set that can be deleted to obtain a disconnected graph. A celebrated theorem of Chvátal and Erdős says that G is Hamiltonian if $\kappa(G) \geq \alpha(G)$. Moreover, Bondy suggested that almost any non-trivial conditions for Hamiltonicity of a graph should also imply pancyclicity. Motivated by this, we prove that if $\kappa(G) \geq 600\alpha(G)$ then G is pancyclic. This establishes a conjecture of Jackson and Ordaz up to a constant factor. Moreover, we obtain the more general result that if G is Hamiltonian with minimum degree $\delta(G) \geq 600\alpha(G)$ then G is pancyclic. Improving an old result of Erdős, we also show that G is pancyclic if it is Hamiltonian and $n \geq 150\alpha(G)^3$. Our arguments use the following theorem of independent interest on cycle lengths in graphs: if $\delta(G) \geq 300\alpha(G)$ then G contains a cycle of length ℓ for all $3 \leq \ell \leq \delta(G)/81$.

1 Introduction

A *Hamilton cycle* is a spanning cycle in a graph, i.e. a cycle passing through all vertices. A graph is called *Hamiltonian* if it contains such a cycle. Hamiltonicity is one of the most fundamental notions in graph theory, tracing its origins to Sir William Rowan Hamilton in the 1850's. Deciding whether a given graph contains a Hamilton cycle is NP-complete, so we do not expect to have a simple characterisation for this property. There is a vast literature in graph theory devoted to obtaining sufficient conditions for Hamiltonicity. For more details we refer the interested reader to the surveys of Gould [11, 12]. The classical result giving such a condition is Dirac's theorem [7], which says that every graph G with $n \geq 3$ vertices and minimum degree at least $n/2$ is Hamiltonian. This theorem was generalised by Bondy [3], who showed that the same assumptions imply that G is *pancyclic*, i.e. contains cycles of length ℓ for all $3 \leq \ell \leq n$. In [4] Bondy proposed the following 'metaconjecture' which has had a considerable influence on research on cycles in graphs.

Metaconjecture. Almost any non-trivial condition on a graph which implies that the graph is Hamiltonian also implies that the graph is pancyclic. (There may be a simple family of exceptional graphs.)

Another classical condition for a graph to be Hamiltonian is given by a theorem of Chvátal and Erdős [6], who showed that if a graph G satisfies $\kappa(G) \geq \alpha(G)$ then it is Hamiltonian. Here $\alpha(G)$ is the *independence number*, i.e. the size of the largest subset of the vertex set that does not contain an edge, and $\kappa(G)$ is the (vertex) *connectivity*, i.e. the size of the smallest subset of the vertex set that can be deleted to obtain a disconnected graph. Motivated by Bondy's metaconjecture, Amar, Fournier and Germa [2] obtained several results on the lengths of cycles in a graph G that satisfies the Chvátal-Erdős condition $\kappa(G) \geq \alpha(G)$. They conjectured that if such a graph G is not bipartite then G contains cycles of length ℓ for all $4 \leq \ell \leq n$. (The case when $G = C_5$ is a 5-cycle needs to be excluded.) Note that the balanced complete bipartite graph $K_{k,k}$ satisfies $\kappa(G) = \alpha(G) = k$ but is not pancyclic, indeed it has no odd cycles. They also made the weaker conjecture that the same conclusion holds under the additional assumption that G is triangle-free. Lou [15] proved the stronger statement that if $\kappa(G) \geq \alpha(G)$ and G is triangle-free then G contains cycles of length ℓ for all $4 \leq \ell \leq n$, unless $G = C_5$ or $G = K_{k,k}$ for some k. Note that the connectivity $\kappa(G)$ is bounded above by the degree of any vertex. If $\kappa(G) > \alpha(G)$ then there is an edge inside any neighbourhood of G, so in particular G must contain triangles. Jackson and Ordaz [14] conjectured that if $\kappa(G) > \alpha(G)$ then G is pancyclic.

To approach these conjectures it is natural to try to prove pancyclicity under a stronger connectivity assumption. A remarkable theorem of Erdős [8], proving a conjecture of Zarins, shows that instead of making a connectivity assumption, it suffices to assume that G is Hamiltonian and the number of vertices is sufficiently large compared to the independence number. He showed that if G is a Hamiltonian graph on n vertices with $\alpha(G) = k$ and $n > 4k^4$ then G is pancyclic. It then follows from [6] that $\kappa(G) \geq 4(\alpha(G) + 1)^4$ is sufficient for pancyclicity. (Various considerably weaker results were subsequently obtained by Flandrin et al., e.g. [10], who were presumably unaware of Erdős' paper.) Our main result improves this bound significantly: we show that a connectivity which is only linear in the independence number suffices for pancyclicity. This establishes the conjecture of Jackson and Ordaz mentioned above up to a constant factor. Moreover, we prove that pancyclicity already follows from assuming that G is Hamiltonian with minimum degree $\delta(G)$ at least linear in the independence number.

Theorem 1.1 *If G is a Hamiltonian graph with $\delta(G) \geq 600\alpha(G)$ then G is pancyclic. In particular, if G is any graph with $\kappa(G) \geq 600\alpha(G)$ then G is pancyclic.*

Erdős [8] remarked that the bound $n > 4k^4$ in his result is unlikely to be best possible. He also noticed that a quadratic lower bound for n in terms of k is necessary for Hamiltonicity to imply pancyclicity. Our next theorem improves Erdős' result and shows that a cubic dependence of n on k is already sufficient.

Theorem 1.2 *If G is a Hamiltonian graph with $|V(G)| \geq 150\alpha(G)^3$ then G is pancyclic.*

Our arguments will use a theorem of independent interest on cycle lengths in graphs.

Theorem 1.3 *If G is a graph with $\delta(G) \geq 300\alpha(G)$ then G contains a cycle of length ℓ for all $3 \leq \ell \leq \delta(G)/81$.*

It is instructive to compare Theorem 1.3 with a result of Nikiforov and Schelp [16], who showed that when the minimum degree $\delta(G)$ is linear in the number of vertices then G contains even cycles of all lengths between 4 and $\delta(G) + 1$ and, after excluding some exceptional cases, odd cycles of all lengths between a constant and $\delta(G) + 1$. We refer the reader to the chapter of Bondy in [5] for other results on cycle lengths in graphs, and to [13, 18, 17] as examples of more recent related results.

Next we describe a simple example showing that Theorems 1.1 and 1.3 are best possible up to the constant factors and that the lower bound for $|V(G)|$ in Theorem 1.2 has to be at least quadratic in $\alpha(G)$. Suppose $k \geq 3$ and let G be the graph on $n = k(2k - 2)$ vertices obtained by taking k vertex-disjoint cliques $X_1, \cdots, X_k$ of size $2k - 2$ and adding a matching of size k which has exactly one edge between X_i and X_{i+1} for all $1 \leq i \leq k$ (where $X_{k+1} := X_1$). Then it is easy to check that G is Hamiltonian, $\alpha(G) = k$ and $\delta(G) = 2k - 3$, but G does not contain a cycle of length $2k - 1$, so is not pancyclic.

The organisation of this paper is as follows. In the next section we collect various known results that we will use in our arguments. We present the proofs of our theorems in Section 3. The final section contains some concluding remarks. We systematically omit rounding signs for the sake of clarity of presentation. We also do not make any serious attempt to optimise absolute constants in our statements and proofs.

Notation. Suppose G is a graph. For a vertex v we let $N(v)$ denote its neighbourhood and $d(v) = |N(v)|$ its degree. If X is a set of vertices then $G[X]$ is the restriction of G to X, i.e. the graph with vertex set X whose edges are edges of G with both endpoints in X. We write $e_G(X) = e(G[X])$ for the number of edges in X. If X and Y are sets of vertices then $e_G(X, Y)$ is the number of edges with one endpoint in X and the other in Y. We omit the subscript G if there is no danger of ambiguity. A walk in G is a sequence of vertices $W = x_0 \cdots x_t$ such that x_i is adjacent to x_{i+1} for $0 \leq i \leq t - 1$. (The vertices need not be distinct.) The length $\ell(W) = t$ of W is the number of edges in W, counting multiplicity of repeated edges. A path is a walk in which no vertices are repeated. A cycle is a walk in which no vertices are repeated, except that the first and last vertices are equal.

2 Preliminaries

In this section we collect various results that will be used in our arguments. We include the short proofs for the convenience of the reader.

2.1 Degrees

We start with two well-known propositions on degrees in graphs.

Proposition 2.1 *Suppose G is a graph with minimum degree at least d. Then G has a bipartite subgraph B with minimum degree at least $d/2$.*

Proof. Consider a bipartite subgraph B of G with as many edges as possible. Let X and Y be the two parts of the bipartition of B. Then any vertex $v \in X$ has at least $d(v)/2$ neighbours in Y, or we could improve the partition by moving v to Y. The same argument applies for $v \in Y$. $\square$

Proposition 2.2 *Suppose G is a graph with average degree at least d. Then G has an induced subgraph with minimum degree at least $d/2$.*

Proof. Suppose G has n vertices and construct a sequence of graphs $G_n = G, G_{n-1}, \cdots$ where if G_i has minimum degree less than $d/2$ we construct G_{i+1} by deleting a vertex of degree less than $d/2$ from G_i. The number of edges deleted in this process is less than $nd/2 \leq e(G)$ so it must terminate at some induced subgraph with minimum degree at least $d/2$. $\square$

2.2 Breadth first search trees

Suppose that B is a graph and x is a vertex of B. We construct a *breadth first search tree T* in B starting at x by the following iterative procedure. We start with T_0 equal to the one-vertex tree on x. Then at step $i \geq 1$, we let N_i be the set of vertices not in T_{i-1} that have at least one neighbour in the tree T_{i-1}, and construct T_i on the vertex set $V(T_{i-1}) \cup N_i$ by adding an edge from each vertex v in N_i to a neighbour of v in T_{i-1}.

Proposition 2.3 *Suppose B is a bipartite graph and T is a breadth first search tree in B starting from a vertex x. Let N_i be the set of vertices at distance i from x in T. Then any vertex in N_i is at distance i from x in B. Also, each N_i is an independent set in B and all edges of B join N_i to N_{i+1} for some $i \geq 0$.*

Now suppose also that B has n vertices and minimum degree $d \geq 5$. Then there is some number $i \geq 0$ such that $e_B(N_i, N_{i+1}) \geq \frac{d}{4}(|N_i| + |N_{i+1}|)$. Furthermore, if $m \geq 0$ is the smallest number with $e_B(N_m, N_{m+1}) \geq \frac{2d}{9}(|N_m| + |N_{m+1}|)$ then $m \geq 1$ and $|N_{i+1}| \geq 2|N_i|$ for $0 \leq i \leq m-1$.

Proof. Let $T_0, T_1, \cdots$ be the sequence of trees in the breadth first search construction. For any v in N_i, the neighbours in B of v within $V(T_{i-1})$ must lie in N_{i-1}, or we would have already added v to T_{i-1}. We deduce that the distance in B from v to x is i. Next consider any y and z in N_i, let P be the path between y and z in T_i and let w be the closest point to x on P. If $w \in N_j$ then the length

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of P is $2(i - j)$, which is even, so yz cannot be an edge, since B is bipartite. This shows that N_i is independent and all edges of B join N_i to N_{i+1} for some $i \geq 0$.

Now suppose that B has n vertices and minimum degree at least d, so that $e(B) \geq dn/2$. We cannot have $e_B(N_i, N_{i+1}) < \frac{d}{4}(|N_i| + |N_{i+1}|)$ for all $i \geq 0$, as this would give the contradiction

$$dn/2 \leq e(B) = \sum_{i \geq 0} e(N_i, N_{i+1}) < \frac{d}{4} \sum_{i \geq 0}(|N_i| + |N_{i+1}|) = \frac{d}{4}(2n - 1).$$

Consider the smallest $m \geq 0$ with $e_B(N_m, N_{m+1}) \geq \frac{2d}{9}(|N_m| + |N_{m+1}|)$. Then $m \geq 1$, since $e_B(N_0, N_1) = |N_1|$ and $d \geq 5$, so $\frac{2d}{9}(|N_0| + |N_1|) \geq |N_1| + 1$. We claim that $|N_{i+1}| \geq 2|N_i|$ for $0 \leq i \leq m - 1$. For suppose that this is not the case, and consider the smallest $0 \leq i \leq m - 1$ for which $|N_{i+1}| < 2|N_i|$. Then $i \geq 1$, since $|N_0| = 1$ and $|N_1| \geq d \geq 5$. There are at least $d|N_i|$ edges incident to N_i, so we must have $e_B(N_{i-1}, N_i) \geq d|N_i|/3$ or $e_B(N_i, N_{i+1}) \geq 2d|N_i|/3$. In the first case we have $e_B(N_{i-1}, N_i) \geq d|N_i|/3 \geq \frac{2d}{9}(|N_{i-1}| + |N_i|)$, since $|N_i| \geq 2|N_{i-1}|$. In the other case we have $e_B(N_i, N_{i+1}) \geq 2d|N_i|/3 \geq \frac{2d}{9}(|N_i| + |N_{i+1}|)$, since $|N_{i+1}| < 2|N_i|$. Either way we have a contradiction to the minimality of m, so the claim is proved. $\qquad\square$

2.3 Independence number

Here we give some well-known relationships between degrees, chromatic number and independence number.

Proposition 2.4 *Suppose G is a graph on n vertices with maximum degree at most k. Then G contains an independent set of size at least $n/(k+1)$.*

Proof. Construct an independent set S greedily by repeatedly choosing any currently available vertex and then marking all its neighbours as unavailable. At the end of this process every vertex of G is either in S or marked as unavailable. At most $k|S|$ vertices have been marked unavailable, so $n \leq |S| + k|S|$, i.e. $|S| \geq n/(k+1)$. $\qquad\square$

Proposition 2.5 *Suppose G is a graph for which every induced subgraph has a vertex of degree at most k. Then G has chromatic number at most $k + 1$.*

Proof. Define a sequence of induced subgraphs $G_n, \cdots, G_0$ starting from $G_n = G$, where G_{i-1} is obtained from G_i by deleting a vertex v_i of degree at most k. Consider the vertices in the order $v_1, \cdots, v_n$ and greedily colour them using $\{1, \cdots, k+1\}$. When we colour v_i we have used at most k colours on its neighbours in G_i, so there is an available colour in $\{1, \cdots, k+1\}$. $\qquad\square$

Proposition 2.6 *Suppose G is a graph with independence number $\alpha(G) \leq k$ and $n \geq dk + 1$ vertices. Then G contains an induced subgraph H with at most $dk + 1$ vertices and minimum degree at least d.*

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Proof. Let S be a set of $dk + 1$ vertices of G. The restriction $G[S]$ of G to S must have chromatic number at least $d + 1$, otherwise it would contain an independent set of size at least $|S|/d > k$, contradicting our assumption on G. By Proposition 2.5 $G[S]$ contains an induced subgraph H with minimum degree at least d. □

2.4   Paths

Now we give some simple tools for manipulating the lengths of paths in cycles when there is a bound on the independence number.

Given any path or cycle W in a graph G, we refer to any set of $t + 1$ consecutive points on W as a *t-interval* (so t is the length of the interval). If J is an interval of length at least 2 such that the endpoints of J are adjacent then we call J a *jump* of W in G. For a jump J we write ∂J for the edge joining the ends of J and J^o for the subinterval of internal points obtained by removing its ends.

Proposition 2.7 *Suppose G is a graph with independence number $\alpha(G) \leq k$, W is a path or cycle in G and I is an interval of length at least $2k$ on W. Then I contains a jump of W in G with length at most $2k$.*

Proof. Starting at one end of I consider the points with positions $1, 3, 5, \cdots, 2k + 1$. This set of $k + 1$ points must contain an edge, since $\alpha(G) \leq k$. □

Proposition 2.8 *Suppose G is a graph with independence number $\alpha(G) \leq k$ and P is a path of length p in G joining two vertices x and y. Then for any number $1 \leq q \leq p$ there is a path of some length ℓ joining x and y with $q \leq \ell \leq q + 2k - 2$.*

Proof. We use induction on p. The statement is clearly true for $p \leq 2k - 1$, so suppose that $p \geq 2k$. By Proposition 2.7 there is a jump J of P in G of some length j with $2 \leq j \leq 2k$. Replacing the portion of P along J by the edge ∂J joining the ends of J gives a path P' of length $p - j + 1$ joining x and y. Now for all $q > p - j + 1$ we can use the original path P, and for all $q \leq p - j + 1$ we can apply the induction hypothesis to P'. □

We also need the following well-known proposition.

Proposition 2.9 *Suppose G is a graph with minimum degree at least d and x is a vertex of G. Then G contains a path of length at least d starting at x. Furthermore, if G is bipartite then G contains such a path of length at least $2d - 1$.*

Proof. Let P be a longest path in G starting at x and let y be the last vertex of P. By the minimum degree condition y has at least d neighbours, and these all belong to P by choice of a longest path, so P contains at least $d + 1$ vertices. Furthermore, if G is bipartite, then y is not adjacent to any vertex at even distance from y along P, so P contains at least $2d$ vertices. □

2.5 Hamiltonicity

Here we give two more substantial lemmas on Hamiltonian graphs which appeared implicitly in [8]. One facilitates absorption of a vertex to create a Hamiltonian graph with one more vertex, the other deletion of a vertex to create a Hamiltonian graph with one fewer vertex.

Lemma 2.10 *Suppose G is a graph, x is a vertex of degree at least $k+1$ in G and $H = G \setminus \{x\}$ is a Hamiltonian graph with independence number $\alpha(H) \leq k$. Then G is Hamiltonian.*

Proof. Suppose that H has n vertices. Label them with $[n] = \{1, \cdots, n\}$ such that $\{i, i+1\}$ is an edge for $1 \leq i \leq n$, where addition is mod n, i.e. $n+1$ is identified with 1. Let $A \subseteq [n]$ be the neighbourhood of x and let $A^+ = \{a+1 : a \in A\}$. Since $|A^+| \geq d(x) \geq k+1 > \alpha(H)$ there is an edge $\{y, z\}$ in A^+, where without loss of generality $y < z$. Now we can form a Hamilton cycle in G by starting at x, going to $z - 1 \in A$, decreasing to y, using the edge $\{y, z\}$ to get to z, increasing to n, going to 1, increasing to $y - 1 \in A$, then ending at x. $\qquad\square$

We remark that the argument in Lemma 2.10 is the main idea in the proof of the Chvátal-Erdős theorem.

Lemma 2.11 *Suppose G is a Hamiltonian graph on $n \geq (2k+1)(k^2+k+1)$ vertices with independence number $\alpha(G) \leq k$. Then G contains a cycle of length $n-1$.*

Proof. Choose a Hamilton cycle C in G and label the vertices as $v_1, \cdots, v_n$ so that the edges of C are $v_i v_{i+1}$ for $1 \leq i \leq n$. (As above we use the convention $v_{n+1} = v_1$.) Set $s = k^2 + k + 1$ and let $I_1, \cdots, I_s$ be disjoint $2k$-intervals in C. Proposition 2.7 gives jumps $J_1, \cdots, J_s$, where each J_i is a subinterval of I_i of length at least 2 and the ends of J_i are adjacent in G. We say that J_i is *good* if each internal vertex $v \in J_i^o$ has at least $k+1$ neighbours in $V(G) \setminus J_i^o$. We claim that there is a good jump. For suppose to the contrary that we can choose $v_i \in J_i^o$ such that v_i has at most k neighbours in $V(G) \setminus J_i^o$ for $1 \leq i \leq s$. Then $\{v_1, \cdots, v_s\}$ spans a subgraph of G with maximum degree at most k, so by Proposition 2.4 contains an independent set of size bigger than k, contradicting our assumption on G. Thus there is a good jump, say J_1. Now we construct a cycle of length $n-1$ as follows. First we replace the portion of C traversing the jump J_1 with the edge ∂J_1 between the endpoints of J_1. Then we use Lemma 2.10 to put back the vertices of J_1^o one by one, increasing the length of the cycle until only one vertex has not been replaced. $\qquad\square$

2.6 Cycles

The following lemma of Erdős, Faudree, Rousseau and Schelp [9] will allow us to find a cycle of some particular length, using a breadth first search tree and the independence assumption. The proof of this lemma can be found in rather abbreviated form within the proof of Theorem 1 in [9]. For the convenience of the reader we include a proof here.

Lemma 2.12 *Suppose G is a graph containing no cycle of length ℓ, T is a tree in G, v is a vertex of T, $h < \ell/2$, and Z is the set of vertices at distance h in T from v. Then the restriction of G to Z is $(\ell-2)$-colourable, and so Z contains a subset of size at least $|Z|/(\ell-2)$ that is independent in G.*

Proof. Fix a plane drawing of T such that in the (x, y)-coordinate system v is at the origin and for $i \geq 0$ points at distance i from v have x-coordinate i. Say that a path $z_0 z_1 \cdots z_t$ in G using vertices of Z is increasing if the y-coordinates of the vertices $z_0, z_1, \cdots, z_t$ forms an increasing sequence.

Given any increasing path $P = z_0 z_1 \cdots z_t$ we let P' be the unique path in T from z_0 to z_t and we let v_P be the closest point to v on P'. We claim that we can remove either z_0 or z_t to obtain a path Q such that $v_Q = v_P$ and Q' has the same length as P'. To see this, we observe that it can only fail if the path in T from z_1 to v_P meets the path from z_t to v_P before it reaches v_P and the path in T from z_{t-1} to v_P meets the path from z_0 to v_P before it reaches v_P. But this would contradict our choice of a plane drawing of T, so the claim holds.

Now we claim that there is no increasing path of length $\ell - 2$. For suppose that $P = z_0 z_1 \cdots z_{\ell-2}$ is an increasing path. Let ℓ' be the length of the path P' in T from z_0 to $z_{\ell-2}$. Then $2 \leq \ell' \leq 2h \leq \ell - 1$. We construct a sequence of paths $P_0 = P, P_1, \cdots, P_{\ell-3}$ where each P_{i+1} is obtained from P_i by removing an endpoint in such a way that $v_{P_{i+1}} = v_{P_i}$, so that for each i we have $v_{P_i} = v_P$ and P_i' has length ℓ'. Since P_i has length $\ell - 2 - i$, $P_i \cup P_i'$ forms a cycle of length $\ell' + \ell - 2 - i$. Setting $i = \ell' - 2$ we obtain a cycle of length ℓ, which contradicts our assumption on G, so the claim holds.

Finally, we define a colouring $c : Z \to \{0, 1, \cdots, \ell - 3\}$ where $c(z)$ is the length of the longest increasing path starting at z. This is a proper colouring of $G[Z]$, as if $z, z' \in Z$ with z below z' (say) then we can add zz' to any increasing path starting at z', so $c(z) > c(z')$. Since all colour classes of c are independent we have an independent set of size at least $|Z|/(\ell-2)$. □

2.7 Probability

Finally we record the standard Chernoff bounds for large deviations of binomial random variables.

Lemma 2.13 *(Chernoff bounds, see [1] Appendix A) Suppose X is a binomial random variable with parameters (n, p) and $a \geq 0$.*

   (i) If $p = 1/2$ then $\mathbb{P}(X - n/2 > a) = \mathbb{P}(X - n/2 < -a) < e^{-2a^2/n}$.

   (ii) $\mathbb{P}(X - np > a) < e^{-a^2/2pn + a^3/2(pn)^2}$.

   (iii) $\mathbb{P}(X - np < -a) < e^{-a^2/2pn}$.

3 Proofs

In this section we present proofs of our three theorems. Throughout we will suppose that G is a graph with independence number $\alpha(G) \leq k$. Also, we can suppose $k \geq 2$, otherwise we have the trivial case

when G is a complete graph. We start with a lemma that provides two vertices that are connected by paths with every length in some interval.

Lemma 3.1 *Suppose G is a graph on n vertices with independence number $\alpha(G) \leq k$ and B is a bipartite subgraph of G with minimum degree $\delta(B) = d > 9k/2$. Suppose x is a vertex of B and let N_i denote the set of vertices at distance i from x in B. Let $m \geq 1$ be the smallest number with $e_B(N_m, N_{m+1}) \geq \frac{2d}{9}(|N_m| + |N_{m+1}|)$. Then*

(i) *$|N_m| \geq 2^{m-1}d$, $m \leq \log_2\left(\frac{n+d-1}{d}\right)$ and G contains cycles of length ℓ for all $3 \leq \ell \leq |N_m|/k$,*

(ii) *there are sets $N_m' \subseteq N_m$ and $N_{m+1}' \subseteq N_{m+1}$ forming the parts of a bipartite subgraph B' of B with minimum degree at least $2d/9$,*

(iii) *there is a vertex y in B' such that there is a path between x and y in G of length ℓ, for any ℓ with $m \leq \ell \leq m + 4d/9 - 2$.*

Proof. By Proposition 2.3 we have $|N_{i+1}| \geq 2|N_i|$ for $0 \leq i \leq m - 1$. Since $|N_1| \geq \delta(B) = d$ we have $|N_i| \geq 2^{i-1}d$ for $1 \leq i \geq m$. Applying Lemma 2.12 to $Z = N_i$ for $1 \leq i \leq m$ we see that G contains cycles of length ℓ for $2i + 1 \leq \ell \leq |N_i|/k$. Since $d > 9k/2$ and $|N_i| \geq 2^{i-1}d$, it is easy to check that $2i + 2 \leq |N_i|/k$, so the intervals $[2i + 1, |N_i|/k], 1 \leq i \leq m$ together contain all integers from 3 to $|N_m|/k$. Also, $n \geq \sum_{i=0}^{m} |N_i| \geq 1 + (2^m - 1)d$ gives the required bound on m, so statement (i) holds. Statement (ii) follows from Proposition 2.2. Indeed, N_m and N_{m+1} form the parts of a bipartite subgraph of B with average degree at least $4d/9$. Thus it contains a subgraph B' with parts $N_m' \subseteq N_m$ and $N_{m+1}' \subseteq N_{m+1}$ with minimum degree at least $d' = 2d/9$. Since $d > 9k/2$ the minimum degree in B' is at least $k + 1$. In particular, $|N_m'| \geq k + 1$. Since $\alpha(G) \leq k$ there is an edge yz of G in N_m'. We claim that this choice of y satisfies statement (iii). To see this we give separate arguments for paths of length $m + 2t$, $t \geq 0$ and paths of length $m + 2t + 1$, $t \geq 0$. By Proposition 2.9, for $0 \leq 2t \leq 2d' - 2 = 4d/9 - 2$ there is a path of length $2t$ in B' from y to a vertex w in N_m', which can be combined with the path in T from w to x to give a path of length $m + 2t$ between x and y. Next, consider the bipartite graph $B' \setminus \{y\}$, which has minimum degree at least $d' - 1$. Then, again by Proposition 2.9, for $0 \leq 2t \leq 2d' - 4$ we can find a path in B' of length $2t$ from z to a vertex $w \in N_m'$, which can be combined with the edge yz and the path in T from w to x to give a path of length $m + 2t + 1$ between x and y. $\square$

Now we prove our first result, which states that a graph G on n vertices with independence number $\alpha(G) \leq k$ and minimum degree $\delta(G) = d \geq 300k$ contains a cycle of length ℓ for all $3 \leq \ell \leq d/81$.

Proof of Theorem 1.3. By Proposition 2.1 we can choose a bipartite subgraph B of G with minimum degree $\delta(B) \geq d/2$. Fix any vertex x and let N_i be the set of vertices at distance i from x in B. By Lemma 3.1, for some $m \geq 1$ we have cycles in G of length ℓ for all $3 \leq \ell \leq |N_m|/k$, where $|N_m| \geq 2^{m-1}(d/2) = 2^{m-2}d$. We also have subsets $N_m' \subseteq N_m$ and $N_{m+1}' \subseteq N_{m+1}$ spanning a bipartite subgraph B' of B with minimum degree at least $\frac{2}{9}(d/2) = d/9$. We can assume that $|N_m| < kd/81$, since otherwise we are done. Also, by choosing $d/9$ neighbours in N_{m+1}' for each vertex

in N'_m and deleting all other vertices of N'_{m+1} we can assume that $|N'_{m+1}| < kd^2/729$. Next we consider a partition $N'_m = P \cup Q$, where each vertex of N'_m is randomly and independently placed in P or Q with probability $1/2$. Since every vertex in N'_{m+1} has degree at least $d/9$, by Chernoff bounds, the probability that there is a vertex in N'_{m+1} having fewer than $d/36$ neighbours in either P or Q is at most $2 \cdot (kd^2/729) \cdot e^{-d/72} < 1$, since $d \geq 300k \geq 600$. Therefore we can choose a partition $N'_m = P \cup Q$ so that every vertex in N'_{m+1} has at least $d/36$ neighbours in P and at least $d/36$ neighbours in Q.

Consider the graph $G^* = G[P \cup N'_{m+1}]$ and its bipartite subgraph B^* with parts P and N'_{m+1}, which has minimum degree $d^* \geq d/36 > 9k/2$. Fix any vertex x^* in P and let N_i^* denote the vertices at distance i from x^* in B^*. By Lemma 3.1, we have some $m^* \geq 1$ such that G^* contains cycles of length ℓ for all $3 \leq \ell \leq |N_{m^*}^*|/k$, where $|N_{m^*}^*| \geq 2^{m^*-1}d^* \geq 2^{m^*}d/72$. We also have a vertex y such that there is a path between x^* and y in G^* of length ℓ, for all ℓ with $m^* \leq \ell \leq m^* + d/81 - 2 \leq m^* + 4d^*/9 - 2$. We let y^* be either equal to y if $y \in P$ or a neighbour of y in Q if $y \in N'_{m+1}$. In either case we have $y^* \in N'_m$ and there are paths between x^* and y^* in the bipartite subgraph B_m of G with parts N_m and N_{m+1} having any length ℓ with $m^* + 1 \leq \ell \leq m^* + d/81 - 2$. Also, x^* and y^* both belong to N_m, so are joined by a path W of some length ℓ_W with $2 \leq \ell_W \leq 2m$, where all internal vertices of W lie in sets N_i with $i < m$. Combining W with paths between x^* and y^* in B_m gives cycles of any length ℓ with $2m + m^* + 1 \leq \ell \leq m^* + d/81$. We already saw that G contains cycles of length ℓ for all $3 \leq \ell \leq \max\{|N_m|, |N_{m^*}^*|\}/k$. Since $d \geq 300k$, we have $|N_m|/k \geq 2^{m-2}d/k > 4m$ and $|N_{m^*}^*|/k \geq 2^{m^*}d/(72k) > 4m^*$. Therefore $\max\{|N_m|, |N_{m^*}^*|\}/k \geq 2m + m^* + 1$, so G contains cycles of length ℓ for all $3 \leq \ell \leq d/81$. $\qquad\square$

Next we need another lemma.

Lemma 3.2 *Suppose G is a graph with independence number $\alpha(G) \leq k$ and $V(G)$ is partitioned into two parts A and B such that*

(i) *$G[A]$ is Hamiltonian,*

(ii) *either $|B| \geq (9k+1)k + 1$ or $G[B]$ has minimum degree at least $9k+1$, and*

(iii) *every vertex in B has at least 2 neighbours in A.*

Then G contains a cycle of length ℓ for any $2k + 1 + \lfloor \log_2(2k+1) \rfloor \leq \ell \leq |A|/2$.

Proof. First we note that $G[B]$ has an induced subgraph H with minimum degree $d \geq 9k + 1$ and at most $(9k+1)k + 1$ vertices. Indeed, if $|B| \leq (9k+1)k + 1$ just take $H = G[B]$, otherwise apply Proposition 2.6 to $G[B]$. By Proposition 2.1, H contains a bipartite subgraph H' with minimum degree $d/2 > 9k/2$. Applying Lemma 3.1 to H and H', we find vertices x and y and a number $m \leq \log_2\left(\frac{|V(H)| + d/2 - 1}{d/2}\right) \leq \log_2(2k+1)$, such that there is a path between x and y in H of length t, for any t with $m \leq t \leq m + 2k - 2 \leq m + \frac{4}{9}(d/2) - 2$. Since every $v \in S$ has at least 2 neighbours in A we can choose neighbours a of x and b of y in A with $a \neq b$. Let P be the path in $G[A]$ joining a and b obtained by taking the longer arc of the Hamilton cycle, so that P has length at least

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$|A|/2$. We construct a cycle of any length ℓ with $2k+1+\log_2(2k+1) \leq \ell \leq |A|/2$ as follows. Since $q = \ell - m - 2k \geq 1$ we can apply Proposition 2.8 to replace P by a path P' in $G[A]$ between a and b of some length ℓ' with $q \leq \ell' \leq q + 2k - 2$. Then $m \leq \ell - 2 - \ell' \leq m + 2k - 2$, so we can complete P' to a cycle of length ℓ by adding the edges ax, by and a path in H of length $\ell - 2 - \ell'$ between x and y. $\qquad\square$

Using this lemma we prove that if G is a Hamiltonian graph on $n \geq 150k^3$ vertices with independence number $\alpha(G) \leq k$ then G is pancyclic.

Proof of Theorem 1.2. Starting from the graph $G = G_n$ we construct a sequence of subgraphs $G_n, G_{n-1}, \cdots, G_{n-20k^2}$, where G_i is a Hamiltonian graph on i vertices. Also, for each removed vertex $v \in V(G) \setminus V(G_i)$ we maintain a set of 2 neighbours $\{a_v, b_v\}$ of v which we never delete, i.e. they will appear in each subgraph of the sequence. To achieve this, consider the graph G_i, let C_i be a Hamilton cycle in G_i and let $N_i = \bigcup \{\{a_v, b_v\} : v \in V(G) \setminus V(G_i)\}$. We claim that we can choose $s = k^2 + k + 1$ disjoint $2k$-intervals in C_i that avoid N_i. To see this, consider the partition of C_i into intervals defined by consecutive points in N_i. We disregard N_i and any intervals of length less than $2k$, then note that we can cover at least half of the remaining points by disjoint $2k$-intervals. Since $|N_i| \leq 40k^2$ and $n \geq 150k^3$ the number of $2k$-intervals thus obtained is at least $\frac{n - 20k^2 - 40k^2(2k+1)}{2(2k+1)} > s$. Now, as in the proof of Lemma 2.11, we can find a good jump J in one of these intervals, and use it to construct a cycle of length $i - 1$. Furthermore, the vertex v removed in this step has at least $k + 1$ neighbours in $V(C_i)$, since it belongs to the good jump J, so we can choose any 2 of these to be a_v and b_v.

This sequence terminates with a Hamiltonian graph $G' = G_{n-20k^2}$ and a set $S = V(G) \setminus V(G')$ of size $20k^2 > (9k+1)k + 1$ such that every $v \in S$ has at least 2 neighbours in $V(G')$. By Lemma 3.2 G contains a cycle of length ℓ for any $2k + 1 + \log_2(2k+1) \leq \ell \leq |V(G')|/2 = n/2 - 10k^2$. To get cycles of length ℓ with $n/2 - 10k^2 \leq \ell \leq n$ we can just repeatedly apply Lemma 2.11 starting from G. To obtain the short cycles, note that $n \geq 150k^3 \geq (300k - 1)k + 1$, since $k \geq 2$, so by Proposition 2.6 G has an induced subgraph G^* with minimum degree $d \geq 300k - 1$. Since $(300k - 1)/81 \geq 2k + 1 + \log_2(2k+1)$, Theorem 1.3 implies that G^* contains cycles of length ℓ for all $3 \leq \ell \leq 2k + 1 + \log_2(2k+1)$. Therefore G is pancyclic. $\qquad\square$

Next we need the following lemma, which provides the long cycles needed in the proof of Theorem 1.1.

Lemma 3.3 *Suppose $k \geq 3$ and G is a Hamiltonian graph on $n \leq 150k^3$ vertices with minimum degree $\delta(G) \geq 600k$ and independence number $\alpha(G) \leq k$. Then G contains a cycle of length ℓ for any $n/12 \leq \ell \leq n$.*

Proof. Consider a partition of the vertices of G into sets X and Y where every vertex is placed randomly and independently in X with probability $1/24$ or in Y with probability $23/24$. By Chernoff bounds, the probability that there is a vertex with less than $25k/2$ neighbours in X is at most $ne^{-25k/8}$ and the probability that X has size more than $n/16$ is at most $e^{-n/384}$. Since $k \geq 3$ and $600k \leq n \leq 150k^3$, both these probabilities are less than 0.4, so we can choose such a partition in which

$|X| \leq n/16$ and every vertex has at least $25k/2$ neighbours in X. Starting from $G_n = G$ we construct a sequence of subgraphs $G_n, G_{n-1}, \cdots, G_{n/12}$, where G_i is a Hamiltonian graph with $|V(G_i)| = i$ and $X \subseteq V(G_i)$. To achieve this, suppose $n/12 < i \leq n$ and C_i is a Hamilton cycle in G_i. We claim that at least $3i/4$ of the vertices of G_i are internal vertices in some jump of length at most $8k$ in C_i. For if this is false, then by averaging we could find an interval I of length $8k$ and a set $S \subseteq I$ of size $2k + 1$ such that no vertex in S is an internal vertex of a jump of length at most $8k$. Consider a subset S' of S of size $k + 1$ formed by taking every other vertex (i.e. the first, the third, ..., the $(2k + 1)^{\text{st}}$). Since $\alpha(G) \leq k$ there must be an edge within S'. This edge forms a jump of length at most $8k$ with at least one internal vertex in S, giving a contradiction which proves the claim. Since $i > n/12$ and $|X| \leq n/16$ we have $|X| < 3i/4$. Thus we can choose a vertex $y \in Y$ and a jump J of length at most $8k$ so that y belongs to J^o (the set of internal vertices of J). Since every vertex has at least $25k/2$ neighbours in X and $X \subseteq V(G_i)$, every vertex in J^o has at least $25k/2 - 8k > k + 1$ neighbours in $V(G_i) \setminus J^o$. We replace the portion of C_i traversing J with the edge ∂J then we use Lemma 2.10 to put back the vertices of J^o one by one, until only y has not been put back. Then $G_{i-1} = G_i \setminus y$ is Hamiltonian with $|V(G_{i-1})| = i - 1$ and $X \subseteq V(G_{i-1})$, as required. $\qquad\square$

Finally we give the proof of our third theorem, which states that if G is a Hamiltonian graph with minimum degree $\delta(G) \geq 600k$ and independence number $\alpha(G) \leq k$ then G is pancyclic.

Proof of Theorem 1.1. Let n be the number of vertices in G. If $n \geq 150k^3$ then we are done by Theorem 1.2 (without even using the minimum degree assumption) so we can suppose that $n < 150k^3$. We also have $n > \delta(G) \geq 600k$, so $k \geq 3$. Applying Theorem 1.3 we see that G has a cycle of length ℓ for all $3 \leq \ell \leq 7k < \delta(G)/81$. Also, by Lemma 3.3 we have a cycle of length ℓ for any $n/12 \leq \ell \leq n$. For the remaining intermediate cycle lengths we consider a partition of the vertices into two sets X and Y, where vertices are randomly and independently placed in X with probability $1/2$ or in Y with probability $1/2$. By Chernoff bounds we can choose this partition so that $|X|, |Y| \geq n/3$ and each vertex has at least $200k$ neighbours in X and at least $200k$ neighbours in Y. Let n' be the smallest number such that there is a subgraph G' of G on n' vertices such that G' is Hamiltonian and $X \subseteq V(G')$. Let C' be a Hamilton cycle in G' and write $D = V(G) \setminus V(G')$. Since $X \subseteq V(G')$ we have $n' \geq n/3$ and $D \subseteq Y$. First we dispose of the case when $|V(G') \cap Y| \leq 4k$. Since every vertex has degree at least $200k$ in Y, the restriction of G to $D = Y \setminus V(G')$ has minimum degree at least $196k$. Every vertex has at least $200k > 1$ neighbours in $X \subseteq V(G')$, so applying Lemma 3.2 with $A = V(G')$ and $B = D$, we obtain a cycle of length ℓ for any $2k + 1 + \log_2(2k + 1) \leq \ell \leq n/6$.

Now we can suppose that $|V(G') \cap Y| \geq 4k + 1$. We can choose an interval I of C' that contains exactly $2k + 1$ vertices of Y and has length at most $n'/2$. Then we consider every other vertex of Y in I to obtain a set of size $k + 1$, which must contain an edge, since $\alpha(G) \leq k$. This gives a jump J of length at most $n'/2$ such that $1 \leq |Y \cap J^o| \leq 2k - 1$. Fix $y_0 \in Y \cap J^o$. We replace the portion of C' traversing J by ∂J, and then use Lemma 2.10 to put back vertices of $J^o \setminus \{y_0\}$ one by one, while we can find such a vertex with at least $k + 1$ neighbours in the current cycle. By minimality of n' this process terminates before all vertices of $J^o \setminus \{y_0\}$ have been replaced. Thus we obtain a non-empty subset S of $J^o \setminus \{y_0\}$ such that $G'' = G' \setminus (S \cup \{y_0\})$ is Hamiltonian and every vertex in S has at most k neighbours

in $V(G'')$. Since J has length at most $n'/2$ and $n' \geq n/3$ we have $n'' = |V(G'')| \geq n/6$. Also, $V(G'')$ contains $X \setminus S$ and every vertex of S has at least $200k$ neighbours in X, of which at most k are in G'', so the restriction $G[S]$ has minimum degree $d \geq 199k$. Choose a vertex $x \in S$ that is adjacent to a vertex a of G''. Such an x exists since $G' = G'' \cup S \cup \{y_0\}$ is Hamiltonian, and in particular 2-connected. By Proposition 2.1 we can choose a bipartite subgraph B of $G[S]$ with minimum degree at least $d/2$. Applying Lemma 3.1 to $G[S]$ and B we obtain a vertex $y \in S$ and a number m such that there is a path between x and y in $G[S]$ of length ℓ, for any ℓ with $m \leq \ell \leq m + 4k$ (say, since $2d/9 - 2 > 4k$), where $m \leq \log_2 \left(\frac{n'/2 + d/2 - 1}{d/2} \right) < \log_2(150k^2/199 + 1) < 2 \log_2 k$.

Let C'' be a Hamilton cycle in G''. If y has a neighbour $b \neq a$ in G'' then we complete the argument as before. We take P to be the longer arc of C'' between a and b, so that P has length at least $n''/2 \geq n/12$. Then we construct a cycle of any length ℓ with $2k + 1 + 2\log_2 k \leq \ell \leq n/12$ as follows. Since $q = \ell - m - 2k \geq 1$ we can apply Proposition 2.8 to replace P by a path P' in G'' between a and b with some length ℓ' with $q \leq \ell' \leq q + 2k - 2$. Then $m \leq \ell - \ell' - 2 \leq m + 2k$, so we can complete P' to a cycle of length ℓ by adding the edges ax, by and a path in $G[S]$ of length $\ell - \ell' - 2$ between x and y. Now suppose that y does not have a neighbour $b \neq a$ in G''. We will repeatedly use the following fact.

($\star$) Any vertex z with at most one neighbour in G'' has at least $40k$ neighbours in D.

The proof of ($\star$) is immediate from that fact that z has at least $200k$ neighbours in Y, but at most $|Y \cap J| + 1 \leq 2k + 2$ of these are in G', so z easily has at least $40k$ neighbours in D. Applying ($\star$) to $z = y$ we can choose a neighbour y' of y in D. Let Z be the connected component of $G[D]$ containing y'. If Z has an induced subgraph Z' with minimum degree at least $20k$ then applying Lemma 3.2 with $A = V(G')$ and $B = V(Z')$ gives a cycle of length ℓ for any $2k + 1 + \log_2(2k + 1) \leq \ell \leq n/6$.

Now suppose that Z does not have any induced subgraph Z' with minimum degree at least $20k$. We claim that there is a path of length at most k in Z from y' to a vertex z in Z with at least 2 neighbours in G''. To see this note first that $|V(Z)| \leq 20k^2$ by Proposition 2.6 and Z contains at least one vertex with at least 2 neighbours in G'' by ($\star$). Now consider a breadth first search tree T in Z starting from y', and for $i \geq 0$ let N_i be the set of vertices at distance i from y' and let Z_i be the restriction of Z to $\cup_{j=0}^{i} N_j$. If every vertex in Z_i has at most one neighbour in G'' then by ($\star$) it has at least $40k$ neighbours in D. On the other hand, we assumed that the minimum degree in $G[Z_i]$ is less than $20k$. Therefore Z_i contains a vertex z with at least $40k - 20k = 20k$ neighbours in $D \setminus V(Z_i)$, implying $|N_{i+1}| \geq 20k$. Since $|V(Z)| \leq 20k^2$ it follows that Z_k contains a vertex z with at least 2 neighbours in G'', as claimed.

Suppose that $z \in N_{i-1}$, where $1 \leq i \leq k + 1$. Combining the paths between x and y in $G[S]$ with the edge yy' and the path of length $i - 1$ in T from y' to z we obtain paths between x and z in $G \setminus V(G'')$ of any length ℓ with $m + i \leq \ell \leq m + 4k + i$, where we recall that $m < 2\log_2 k$. Choose a neighbour $b \neq a$ of z in G'' and let P be the longer arc of C'' between a and b, so that P has length at least $n''/2 \geq n/12$. (See Figure 1.) Now we construct a cycle of any length ℓ with $3k + 2 + 2\log_2 k \leq \ell \leq n/12$ as follows. Since $q = \ell - m - 2k - i \geq 1$ we can apply Proposition 2.8 to replace P by a path P' in G'' between a and b with some length ℓ' with $q \leq \ell' \leq q + 2k - 2$. Then
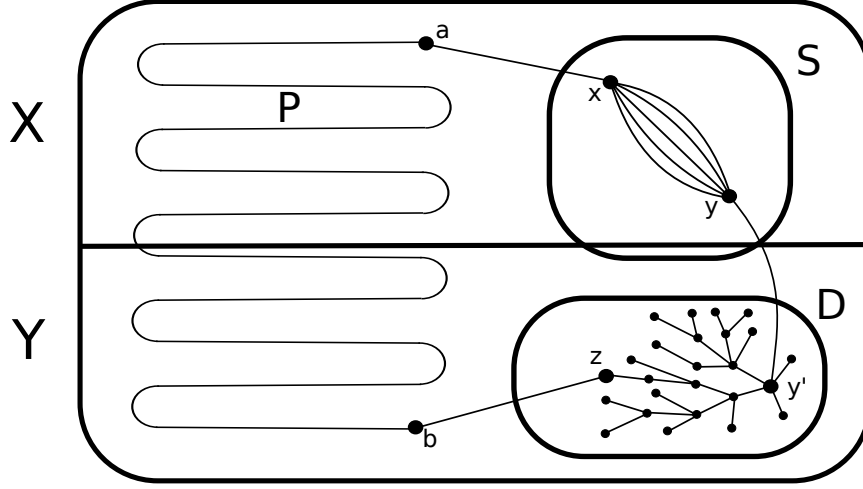
Figure 1: Constructing cycles of intermediate length

$m + i \leq \ell - \ell' - 2 \leq m + 2k + i$, so we can complete P' to a cycle of length ℓ by adding the edges ax, bz and a path in $G[S]$ of length $\ell - \ell' - 2$ between x and z.

Since $2k + 1 + 2\log_2 k < 3k + 2 + 2\log_2 k < 7k$, in all cases we find cycles of length ℓ for $7k \leq \ell \leq n/12$. Recall that we also have cycles of length ℓ when $3 \leq \ell \leq 7k$ and when $n/12 \leq \ell \leq n$. This implies pancyclicity of G. □

4   Concluding remarks

We have answered the question of Jackson and Ordaz up to a constant factor. Obviously it would be nice to obtain the exact bound, but perhaps one should first attempt to prove an asymptotic version, i.e. that if $\kappa(G) \geq (1 + o(1))\alpha(G)$ then G is pancyclic. Also, it would be interesting to give the correct order of magnitude for the minimum number n of vertices such that any Hamiltonian graph G on n vertices with $\alpha(G) = k$ is pancyclic. We proved that this holds if $n = \Omega(k^3)$, but it probably can be reduced to $n = \Omega(k^2)$. One way to attack this problem is to improve the estimate in Lemma 2.11, which says that any Hamiltonian graph with independence number k and $n = \Omega(k^3)$ vertices contains a cycle of length $n - 1$. It would be extremely interesting to determine the correct dependence of n on k for this problem of just removing one vertex. Even the following question remains open.

Question. Is there an absolute constant C such that any Hamiltonian graph with independence number k and $n \geq Ck$ vertices contains a cycle of length $n - 1$?

A positive answer would be tight up to a constant factor (clearly) and in combination with Proposition 2.6 and Theorem 1.3 would immediately imply that a quadratic dependence of n on k is sufficient for Hamiltonicity to imply pancyclicity.

References

[1] N. Alon and J. Spencer, *The probabilistic method*, second edition, Wiley, New York, 2000.

[2] D. Amar, I. Fournier and A. Germa, Pancyclism in Chvátal-Erdős's graphs, *Graphs Combin.* **7** (1991), 101–112.

[3] J. A. Bondy, Pancyclic graphs I, *J. Combin. Theory Ser. B* **11** (1971), 80–84.

[4] J. A. Bondy, Pancyclic graphs: recent results, Infinite and finite sets, *Colloq. Math. Soc. János Bolyai*, 181–187, Keszthely, Hungary, 1973.

[5] J. A. Bondy, Basic graph theory: paths and circuits, Handbook of Combinatorics, 3–110, North-Holland, Amsterdam, 1995.

[6] V. Chvátal and P. Erdős, A note on Hamiltonian circuits, *Discrete Math.* **2** (1972), 111–113.

[7] G. A. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc.* **2** (1952), 69–81.

[8] P. Erdős, Some problems in graph theory, Hypergraph Seminar (Ohio State Univ., Columbus, Ohio, 1972), *Lecture Notes in Math.* **411**, 187–190, Springer, Berlin, 1974.

[9] P. Erdős, R. J. Faudree, C. C. Rousseau and R. H. Schelp, On cycle–complete graph Ramsey numbers, *J. Graph Theory* **2** (1978), 53–64.

[10] E. Flandrin, H. Li, A. Marczyk and M. Wozniak, A note on pancyclism of highly connected graphs, *Disc. Math.* **286** (2004), 57–60.

[11] R. J. Gould, Updating the hamiltonian problem - a survey, *J. Graph Theory* **15** (1991), 121–157.

[12] R. J. Gould, Advances on the hamiltonian problem - a survey, *Graphs Combin.* **19** (2003), 7–52.

[13] R. J. Gould, P. E. Haxell and A. D. Scott, A note on cycle lengths in graphs, *Graphs Combin.* **18** (2002), 491–498.

[14] B. Jackson and O. Ordaz, Chvátal-Erdős conditions for paths and cycles in graphs and digraphs: a survey, *Disc. Math.* **84** (1990), 241–254.

[15] D. Lou, The Chvátal-Erdős condition for cycles in triangle-free graphs, *Disc. Math.* **152** (1996), 253–257.

[16] V. Nikiforov and R. H. Schelp, Cycle lengths in graphs with large minimum degree, *J. Graph Theory* **52** (2006), 157–170.

[17] B. Sudakov and J. Verstraëte, Cycle lengths in sparse graphs, *Combinatorica* **28** (2008), 357–372.

[18] J. Verstraëte, On arithmetic progressions of cycle lengths in graphs, *Combin. Probab. Comput.* **9** (2000), 369–373.